

# M10 Simple Models of Rates of Return

## Topics in Insurance, Risk, and Finance

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2026



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# Learning outcomes

- Define deterministic model and stochastic model and understand their pros and cons.
- Define fixed rate model and varying rate model.
- Understand notations  $S_n$ ,  $V_n$ ,  $A_n$ , and  $P_n$ .
- Calculate or derive an analytical expression of moments (especially the first two) of series of cash flows, not limited to  $S_n$ ,  $V_n$ ,  $A_n$ , and  $P_n$ .
- Derive algebraically recursive relationships between cash flows.

# Investment rates

- Many financial contracts are long-term (e.g., annuities), and interest rates/investment rates play an important role in determining the values of these contracts.
- In many contexts, rates of return are regarded as constant, mainly for mathematical convenience.
- However, interest rates/investment rates can change drastically due to the changes of economic environment.
- Rates of return are essentially random/stochastic.

# Constant or random?

Q: What is wrong with models with constant rates of return?

- Denote by  $i$  the rate of return, which is a random variable.
- If we want to pick a number as the constant rate of return, a natural candidate is the mean rate of return, i.e.,  $\mathbb{E}(i)$ .
- Let us compute the accumulated value of 1 and the present value of 1 using constant and random rates of returns, respectively.
- Jensen's inequality: if  $X$  is a random variable and  $f$  is a convex function, then  $\mathbb{E}(f(X)) \geq f(\mathbb{E}(X))$ .

# Investment rates

- Using constant rates of return may underestimate risks.
- Modelling investment rates is a crucial task for financial institutions.
- We will study simple stochastic models for rates of return in this subject.
- Advanced models will be covered in another subject.

# $S_n$ and $V_n$

Assume that the rate of return in period  $[t - 1, t]$  is  $i_t$ , then

- A single investment of 1 at time 0 will accumulate at time  $n$  to:

$$S_n = \prod_{t=1}^n (1 + i_t).$$

- The present value of 1 at time  $n$  is worth at time 0:

$$V_n = \prod_{t=1}^n (1 + i_t)^{-1}.$$

## $A_n$ and $P_n$

- A series of annual investments, each of amount 1, at times  $0, 1, \dots, n - 1$  will accumulate at time  $n$  to:

$$A_n = \sum_{t=1}^n \prod_{k=t}^n (1 + i_k).$$

- The present value of a series of annual investments, each of amount 1, at times  $1, 2, \dots, n$  is:

$$P_n = \sum_{t=1}^n \prod_{k=1}^t (1 + i_k)^{-1}.$$

# Objectives

We will study the above quantities using:

- Simple models for the rates of return,
- Lognormal distributions to model the rates of return,
- Numerical simulations in Excel.

## Two types of models: deterministic model

Deterministic model: The investment rates are deterministic over the time period (note the difference between “deterministic” and “constant”). Pros:

- easy to use
- mathematically friendly

Cons:

- hard to determine the rates
- provides one single answer which is correct only if the rates are correct
- not useful for risk management

Note that in a deterministic model, the interest rates are prespecified, hence the aforementioned quantities are constants.

## Two types of models: stochastic model

Stochastic model: The investment rates are allowed to vary with the use of probability/statistics methods. Pros:

- allow us to have a range of answers to our problem
- good for risk management (variance can be useful)

Cons:

- calculation is more challenging
- sometimes is not mathematically friendly
- models can be wrong

Note that in a stochastic model, the aforementioned quantities are random variables.

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# Fixed rate model

In a fixed rate model

- the investment rate follows a distribution function
- it is determined right after the investment is made
- the rate will be a constant throughout the period of the investment
- the rate is applied as a compound rate
- Note that it belongs to the class of stochastic models.

## Fixed rate model: example A I

Suppose that an investment of 5000 is made and will be accumulate at the investment rate  $i_k$ . The annual investment rate  $i$  in a fixed rate model follows a three-point distribution:  $\mathbb{P}(i = 0.06) = 0.2$ ,  $\mathbb{P}(i = 0.08) = 0.7$ , and  $\mathbb{P}(i = 0.10) = 0.1$ . Consider the following questions.

- Find the expectation and variance of the accumulated investment value at the end of year 5.
- Find the accumulated investment value at the mean rate of return.

# Fixed rate model: example A II

# Fixed rate model: example A III

## Fixed rate model: example B I

Consider events  $A$  and  $A^c$  which denote the two states of an economy (e.g., good and bad) such that  $\mathbb{P}(A) = 0.5$ . In state  $A$ , the rate of return is uniformly distributed on  $0.1$  and  $0.2$ . In state  $A^c$ , the rate of return is uniformly distributed on  $0.3$  and  $0.4$ . Find the mean of  $S_2$ .

# Fixed rate model: example B II

## Fixed rate model: example C

Suppose the rate of return  $i$  follows the distribution

$$F(x) = \begin{cases} 0, & 0 \leq x < 0.2, \\ x/0.3, & 0.2 \leq x \leq 0.3. \end{cases}$$

Find the mean of  $V_1$ .

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# Varying rate model

In a varying rate model,

- each investment rate follows a distribution
- it is determined for each period of investment
- the rate of return for each period will be one realization from the distribution
- the rates in different periods are independent of each other
- the rates are applied as compound rates

Key difference: the rates in a fixed rate model is constant throughout the life of the investment while those in a varying rate model can be different in each period of investments.

## Varying rate model: example

Assume that the rate of return for each period takes values 0.02, 0.04, and 0.06 with equal probabilities. What is the probability that

$$S_n = 1.02 * 1.06^{n-1}?$$

# Varying rate model

Consider a time interval  $[0, n]$  with subintervals  $[0, 1], \dots, [n - 1, n]$ . We fix the following notations:

- $P_t$ : the investment made at time  $t$
- $F_t$ : the accumulated value of the investment right before time  $t$  (i.e., before any investment is made at time  $t$ )

Then we have the recursive relation for  $t = 1, \dots, n$ :

$$F_t = (1 + i_t)(F_{t-1} + P_{t-1})$$

## Varying rate model: a simple example

Suppose that  $F_0 = 0$ ,  $P_0 = 1$ ,  $P_2 = 2$  and  $\mathbb{E}[i_t] = 0.1$ . Find the mean of accumulated value at time 3.

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# Moments of $S_n$

## Proposition

*In the varying rate model, the  $k$ th moments of  $S_n$  and  $V_n$  are*

$$\mathbb{E}[S_n^k] = \prod_{t=1}^n \mathbb{E}[(1 + i_t)^k] \quad \text{and} \quad \mathbb{E}[V_n^k] = \prod_{t=1}^n \mathbb{E}[(1 + i_t)^{-k}].$$

Note that  $\mathbb{E}[V_n] \neq \mathbb{E}[S_n]^{-1}$

## Moments of $S_n$ : example A

Suppose that each of  $i_1, \dots, i_n$  has mean  $\mu$  and variance  $\sigma^2$ . Find the expectation and variance of  $S_n$ .

## Moments of $S_n$ : example B

In a varying rate model, let  $i_1, \dots, i_n$  have the following distribution:

$$i_t = \begin{cases} 0.04 & \text{with probability } 0.25, \\ 0.06 & \text{with probability } 0.60, \\ 0.08 & \text{with probability } 0.15. \end{cases}$$

Calculate the mean and variance of  $S_n$  for  $n = 5, 10, 20$ , and comment on the values.

We have

$$\mathbb{E}(S_5) = 1.3256, \mathbb{E}(S_{10}) = 1.7573, \mathbb{E}(S_{20}) = 3.0883,$$

and

$$\text{Var}(S_5) = 0.0012, \text{Var}(S_{10}) = 0.0045, \text{Var}(S_{20}) = 0.0266.$$



## Moments of $S_n$ : example B

As  $n$  increases:

- The expected accumulation increases. This is as we would expect since the longer the period of the accumulation, the greater the accumulation, and its expected value, must be.
- The variance of the accumulation increases. That is, the longer the period into the future, the more uncertain we are about what the accumulation will be.

## Moments of $V_n$ : example A

In a varying rate model, let  $i_1, \dots, i_n$  have the following distribution:

$$i_t = \begin{cases} 0.04 & \text{with probability } 0.25, \\ 0.06 & \text{with probability } 0.60, \\ 0.08 & \text{with probability } 0.15. \end{cases}$$

Calculate the mean and variance of  $V_n$  for  $n = 5, 10, 20$ , and comment on the values.

We have

$$\mathbb{E}(V_5) = 0.7549, \mathbb{E}(V_{10}) = 0.5698, \mathbb{E}(V_{20}) = 0.3247,$$

and

$$\text{Var}(V_5) = 0.00040, \text{Var}(V_{10}) = 0.00045, \text{Var}(V_{20}) = 0.00029.$$



## Moments of $V_n$ : example A

As  $n$  increases:

- The expected discounted value decreases. This is as we would expect.
- The variance of the discounted value decreases. Does it mean that, the longer the period into the future, the less uncertain we are about what the discounted value will be?
- Another way to measure variability is the coefficient of variation ( $cv$ ): For a random variable  $X$ , we have

$$cv(X) = \frac{\text{Sd}(X)}{\mathbb{E}(X)}.$$

- We have  $cv(V_5) = 0.0264$ ,  $cv(V_{10}) = 0.0373$ ,  $cv(V_{20}) = 0.0528$ , which increase as  $n$  increases. Hence, we have more uncertainty in this perspective.

## Moments of $V_n$ : example B

Assume that  $i_1, \dots, i_n$  are iid and  $i_t \sim U(0.05, 0.09)$  for  $t = 1, \dots, n$ . Find the mean and variance of the present value of a unit payment in 20 periods.

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# Recursive formula

- Unlike  $S_n$  and  $V_n$ , it is generally not easy to derive analytical expressions for the moments of  $A_n$  and  $P_n$ .
- For this, we can use recursive formulas of  $A_n$  and  $P_n$  to derive their moments.

## Moments of $A_n$ : mean

### Proposition

*In the varying rate model, if  $i_1, \dots, i_n$  have the same mean  $\mu$ , then*

$$\mathbb{E}[A_n] = \frac{(1 + \mu)^n - 1}{d},$$

*where  $d = \mu/(1 + \mu)$ .*

## Moments of $A_n$ : second moment

Since

$$A_n^2 = (1 + i_n)^2(A_{n-1} + 1)^2,$$

and  $i_1, \dots, i_n$  are independent, we have

$$\mathbb{E}[A_n^2] = \mathbb{E}[(i_n^2 + 2i_n + 1)]\mathbb{E}[(A_{n-1}^2 + 2A_{n-1} + 1)].$$

If  $i_1, \dots, i_n$  have the same mean  $\mu$  and variance  $\sigma^2$ , we get a recursive relation

$$\mathbb{E}[A_n^2] = (1 + 2\mu + \mu^2 + \sigma^2)(\mathbb{E}[A_{n-1}^2] + 2\mathbb{E}[A_{n-1}] + 1),$$

where  $\mathbb{E}[A_{n-1}]$  is known.

## Moments of $A_n$ : example

Suppose that in a varying rate model  $i_1, \dots, i_n$  follow a uniform distribution on  $[0.02, 0.06]$ . Find the mean and variance of  $A_5$ .

We first note that for  $t = 1, \dots, n$ ,

$$\mu = \mathbb{E}[i_t] = \frac{0.02 + 0.06}{2} = 0.04,$$

and

$$\sigma^2 = \text{Var}(i_t) = \frac{(0.06 - 0.02)^2}{12} \approx 0.00013333.$$

The mean of  $A_5$  is

$$\mathbb{E}[A_5] = \frac{(1 + \mu)^5 - 1}{d} = \frac{1.04^5 - 1}{0.04/1.04} = 5.63298.$$

## Moments of $A_n$ : example

To get the variance of  $A_5$ , we need to use the recursive formula

$$\begin{aligned}\mathbb{E}[A_n^2] &= (1 + 2\mu + \mu^2 + \sigma^2)(\mathbb{E}[A_{n-1}^2] + 2\mathbb{E}[A_{n-1}] + 1) \\ &= 1.08173333(\mathbb{E}[A_{n-1}^2] + 2\mathbb{E}[A_{n-1}] + 1).\end{aligned}$$

We first compute

$$\mathbb{E}[A_1] = 1.04, \mathbb{E}[A_2] = 2.1216, \mathbb{E}[A_3] = 3.24646,$$

$$\mathbb{E}[A_4] = 4.41632.$$

## Moments of $S_n$ and $A_n$ : example IV

Using the recursive relation, we get

$$\mathbb{E}[A_1^2] = 1.08173, \mathbb{E}[A_2^2] = 4.50189, \mathbb{E}[A_3^2] = 10.54158,$$

$$\mathbb{E}[A_4^2] = 19.50853, \mathbb{E}[A_5^2] = 31.73933.$$

Hence, the variance is

$$\text{Var}(A_5) = 31.73933 - 5.63298^2 = 0.0089172.$$

## Moments for $P_n$ I

We can use similar techniques to derive moments for  $P_n$  as well. We have

$$\begin{aligned}P_n &= \sum_{t=1}^n \prod_{k=1}^t (1 + i_k)^{-1} \\&= \prod_{k=1}^1 (1 + i_k)^{-1} + \prod_{k=1}^2 (1 + i_k)^{-1} + \dots + \prod_{k=1}^n (1 + i_k)^{-1} \\&= (1 + i_1)^{-1} \left( 1 + (1 + i_2)^{-1} + \dots + \prod_{k=2}^n (1 + i_k)^{-1} \right) \\&= (1 + i_1)^{-1} (1 + P_{n-1}^*),\end{aligned}$$

where  $P_{n-1}^*$  is the time-1 value of payments of 1 at times 2, ...,  $n$ .

## Moments for $P_n$ II

Note that  $P_{n-1}^*$  and  $P_{n-1}$  have the same distribution for iid rates. Hence,

$$\begin{aligned}\mathbb{E}(P_n) &= \mathbb{E}((1 + i_1)^{-1}(1 + P_{n-1}^*)) \\ &= \mathbb{E}((1 + i_1)^{-1})(1 + \mathbb{E}(P_{n-1}^*)) \\ &= \mathbb{E}((1 + i_1)^{-1})(1 + \mathbb{E}(P_{n-1})),\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}(P_n^2) &= \mathbb{E}((1 + i_1)^{-2}(1 + P_{n-1}^*)^2) \\ &= \mathbb{E}((1 + i_1)^{-2})(1 + 2\mathbb{E}(P_{n-1}) + \mathbb{E}(P_{n-1}^2)).\end{aligned}$$

# Infinite series of payments

Suppose that returns  $i_1, i_2, i_3 \dots$  are iid. The time-0 value of payments of 1 at times  $0, 1, 2, \dots$  is denoted by  $c_\infty$ . Find the first two moments of  $c_\infty$ .

# Infinite series of payments



## Moments for $P_n$ : analytical solutions

Suppose  $i_1, \dots, i_n$  are iid. Denote by  $u_k$  the  $k$ th moment  $\mathbb{E}((1 + i_t)^{-k})$ . Find the moments of  $P_n$ .

We have

$$\begin{aligned}\mathbb{E}(P_n) &= \mathbb{E}\left(\sum_{t=1}^n \prod_{k=1}^t (1 + i_k)^{-1}\right) \\ &= \sum_{t=1}^n \prod_{k=1}^t \mathbb{E}((1 + i_k)^{-1}) \\ &= \sum_{t=1}^n u_1^t = \frac{u_1(1 - u_1^n)}{1 - u_1}.\end{aligned}$$

## Moments for $P_n$ : analytical solutions

Finding the second moment is more difficult. We look at the case when  $n = 2$ : We have

$$\begin{aligned}\mathbb{E}(P_2^2) &= \mathbb{E}\left(\left((1+i_1)^{-1} + (1+i_1)^{-1}(1+i_2)^{-1}\right)^2\right) \\ &= \mathbb{E}\left((1+i_1)^{-2}\right) + 2\mathbb{E}\left((1+i_1)^{-2}\right)\mathbb{E}\left((1+i_2)^{-1}\right) \\ &\quad + \mathbb{E}\left((1+i_1)^{-2}\right)\mathbb{E}\left((1+i_2)^{-2}\right) \\ &= u_2 + 2u_1u_2 + u_2^2.\end{aligned}$$

## Moments for $P_n$ : analytical solutions

Next, we find the second moment of  $P_n$ . We have

$$\begin{aligned}\mathbb{E}(P_n^2) &= \mathbb{E} \left( \left( \sum_{t=1}^n \prod_{k=1}^t (1 + i_k)^{-1} \right)^2 \right) \\ &= \mathbb{E} \left( \left( \sum_{t=1}^n V_t \right)^2 \right) \\ &= \mathbb{E} \left( \sum_{t=1}^n V_t^2 + 2 \sum_{j < k} V_j V_k \right) \\ &= \sum_{t=1}^n \mathbb{E}(V_t^2) + 2 \sum_{j < k} \mathbb{E}(V_j V_k).\end{aligned}$$

We know

$$\sum_{t=1}^n \mathbb{E}(V_t^2) = \sum_{t=1}^n u_2^t = \frac{u_2(1 - u_2^n)}{1 - u_2}.$$

## Moments for $P_n$ : analytical solutions

The rest is to find the expectation of the cross term. For  $j < k$ , we have

## Moments for $P_n$ : analytical solutions

Then

$$\begin{aligned}\sum_{j < k} \mathbb{E}(V_j V_k) &= \sum_{j=1}^{n-1} \sum_{k=j+1}^n u_2^j u_1^{k-j} \\ &= \sum_{j=1}^{n-1} \left(\frac{u_2}{u_1}\right)^j \sum_{k=j+1}^n u_1^k \\ &= \sum_{j=1}^{n-1} \left(\frac{u_2}{u_1}\right)^j \frac{u_1^{j+1} - u_1^{n+1}}{1 - u_1} \\ &= \frac{u_1}{1 - u_1} \left( \sum_{j=1}^{n-1} u_2^j - u_1^n \sum_{j=1}^{n-1} \left(\frac{u_2}{u_1}\right)^j \right) \\ &= \frac{u_1 u_2}{1 - u_1} \left( \frac{1 - u_2^{n-1}}{1 - u_2} - u_1 \frac{u_1^{n-1} - u_2^{n-1}}{u_1 - u_2} \right).\end{aligned}$$