M10 Simple Models of Rates of Return

Topics in Insurance, Risk, and Finance

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Introduction

Learning outcomes

- Define deterministic model and stochastic model and understand their pros and cons.
- Define fixed rate model and varying rate model.
- Understand notations S_n, V_n, A_n, and P_n.
- Calculate or derive an analytical expression of moments (especially the first two) of series of cash flows, not limited to S_n, V_n, A_n, and P_n.
- Derive algebraically recursive relationships between cash flows.

Investment rates

- Many financial contracts are long-term (e.g., annuities), and interest rates/investment rates play an important role in determining the values of these contracts.
- In many contexts, rates of return are regarded as constant, mainly for mathematical convenience.
- However, interest rates/investment rates can change drastically due to the changes of economic environment.
- Rates of return are essentially random/stochastic.

Constant or random?

Q: What is wrong with models with constant rates of return?

- Denote by i the rate of return, which is a random variable.
- If we want to pick a number as the constant rate of return, a natural candidate is the mean rate of return, i.e., $\mathbb{E}(i)$.
- Let us compute the accumulated value of 1 and the present value of 1 using constant and random rates of returns, respectively.
- Jensen's inequality: if X is a random variable and f is a convex function, then $\mathbb{E}(f(X)) \geqslant f(\mathbb{E}(X))$.

Investment rates

- Using constant rates of return may underestimate risks.
- Modelling investment rates is a crucial task for financial institutions.
- We will study simple stochastic models for rates of return in this subject.
- Advanced models will be covered in another subject.

S_n and V_n

Assume that the rate of return in period $[t-1, t]$ is i_t, then

- A single investment of 1 at time 0 will accumulate at time n to:

$$S_n = \prod_{t=1}^{n}(1 + i_t).$$

- The present value of 1 at time n is worth at time 0:

$$V_n = \prod_{t=1}^{n} (1 + i_t)^{-1}.$$

A_n and P_n

- A series of annual investments, each of amount 1, at times $0, 1, \ldots, n-1$ will accumulate at time n to:

$$A_n = \sum_{t=1}^{n} \prod_{k=t}^{n} (1 + i_k).$$

- The present value of a series of annual investments, each of amount 1, at times $1, 2, \ldots, n$ is:

$$P_n = \sum_{t=1}^{n} \prod_{k=1}^{t} (1 + i_k)^{-1}.$$

Objectives

We will study the above quantities using:

- Simple models for the rates of return,
- Lognormal distributions to model the rates of return,
- Numerical simulations in Excel.

Two types of models: deterministic model

Deterministic model: The investment rates are deterministic over the time period (note the difference between "deterministic" and "constant"). Pros:

- easy to use
- mathematically friendly

Cons:

- hard to determine the rates
- provides one single answer which is correct only if the rates are correct
- not useful for risk management

Note that in a deterministic model, the interest rates are prespecified, hence the aforementioned quantities are constants.

Two types of models: stochastic model

Stochastic model: The investment rates are allowed to vary with the use of probability/statistics methods. Pros:

- allow us to have a range of answers to our problem

- good for risk management (variance can be useful)

Cons:

- calculation is more challenging
- sometimes is not mathematically friendly
- models can be wrong

Note that in a stochastic model, the aforementioned quantities are random variables.

Fixed rate model

Fixed rate model

In a fixed rate model

- the investment rate follows a distribution function
- it is determined right after the investment is made
- the rate will be a constant throughout the period of the investment
- the rate is applied as a compound rate
- Note that it belongs to the class of stochastic models.

Fixed rate model: example A I

Suppose that an investment of 5000 is made and will be accumulate at the investment rate i_k. The annual investment rate i in a fixed rate model follows a three-point distribution: $\mathbb{P}(i = 0.06) = 0.2$, $\mathbb{P}(i = 0.08) = 0.7$, and $\mathbb{P}(i = 0.10) = 0.1$. Consider the following questions.

- Find the expectation and variance of the accumulated investment value at the end of year 5.
- Find the accumulated investment value at the mean rate of return.

Fixed rate model: example A II

Fixed rate model: example A III

Fixed rate model: example B I

Consider events A and A^c which denote the two states of an economy (e.g., good and bad) such that $\mathbb{P}(A) = 0.5$. In state A, the rate of return is uniformly distributed on 0.1 and 0.2. In state A^c, the rate of return is uniformly distributed on 0.3 and 0.4. Find the mean of S_2.

Fixed rate model: example B II

Fixed rate model: example C

Suppose the rate of return i follows the distribution

$$F(x) = \begin{cases} 0, \quad 0 \leqslant x < 0.2, \\ x/0.3, \quad 0.2 \leqslant x \leqslant 0.3. \end{cases}$$

Find the mean of V_1.

Varying rate model

Varying rate model

In a varying rate model,

- each investment rate follows a distribution
- it is determined for each period of investment
- the rate of return for each period will be one realization from the distribution
- the rates in different periods are independent of each other
- the rates are applied as compound rates

Key difference: the rates in a fixed rate model is constant throughout the life of the investment while those in a varying rate model can be different in each period of investments.

Varying rate model: example

Assume that the rate of return for each period takes values 0.02, 0.04, and 0.06 with equal probabilities. What is the probability that $S_n = 1.02 * 1.06^{n-1}$?

Varying rate model

Consider a time interval $[0, n]$ with subintervals $[0, 1], \dots, [n - 1, n]$. We fix the following notations:

- P_t: the investment made at time t
- F_t: the accumulated value of the investment right before time t (i.e., before any investment is made at time t)

Then we have the recursive relation for $t = 1, \dots, n$:

$$F_t = (1 + i_t)(F_{t-1} + P_{t-1})$$

Varying rate model: a simple example

Suppose that $F_0 = 0$, $P_0 = 1$, $P_2 = 2$ and $\mathbb{E}[i_t] = 0.1$. Find the mean of accumulated value at time 3.

Varying rate model: S_n and V_n

Moments of S_n

Proposition 1. *In the varying rate model, the kth moments of S_n and V_n are*

$$\mathbb{E}[S_n^k] = \prod_{t=1}^{n} \mathbb{E}\left[(1+i_t)^k\right] \quad \textit{and} \quad \mathbb{E}[V_n^k] = \prod_{t=1}^{n} \mathbb{E}\left[(1+i_t)^{-k}\right].$$

Note that $\mathbb{E}[V_n] \neq \mathbb{E}[S_n]^{-1}$

Moments of S_n: example A

Suppose that each of $i_1, \ldots, i_n$ has mean μ and variance σ^2. Find the expectation and variance of S_n.

Moments of S_n: example B

In a varying rate model, let $i_1, \ldots, i_n$ have the following distribution:

$$i_t = \begin{cases} 0.04 \text{ with probability } 0.25, \\ 0.06 \text{ with probability } 0.60, \\ 0.08 \text{ with probability } 0.15. \end{cases}$$

Calculate the mean and variance of S_n for $n = 5, 10, 20$, and comment on the values.

We have

$$\mathbb{E}(S_5) = 1.3256, \mathbb{E}(S_{10}) = 1.7573, \mathbb{E}(S_{20}) = 3.0883,$$

and

$$\mathrm{Var}(S_5) = 0.0012, \mathrm{Var}(S_{10}) = 0.0045, \mathrm{Var}(S_{20}) = 0.0266.$$

Moments of S_n: example B

As n increases:

- The expected accumulation increases. This is as we would expect since the longer the period of the accumulation, the greater the accumulation, and its expected value, must be.
- The variance of the accumulation increases. That is, the longer the period into the future, the more uncertain we are about what the accumulation will be.

Moments of V_n: example A

In a varying rate model, let $i_1, \ldots, i_n$ have the following distribution:

$$i_t = \begin{cases} 0.04 \text{ with probability } 0.25, \\ 0.06 \text{ with probability } 0.60, \\ 0.08 \text{ with probability } 0.15. \end{cases}$$

Calculate the mean and variance of V_n for $n = 5, 10, 20$, and comment on the values.

We have

$$\mathbb{E}(V_5) = 0.7549, \mathbb{E}(V_{10}) = 0.5698, \mathbb{E}(V_{20}) = 0.3247,$$

and

$$\mathrm{Var}(V_5) = 0.00040, \mathrm{Var}(V_{10}) = 0.00045, \mathrm{Var}(V_{20}) = 0.00029.$$

Moments of V_n: example A

As n increases:

- The expected discounted value decreases. This is as we would expect.
- The variance of the discounted value decreases. Does it mean that, the longer the period into the future, the less uncertain we are about what the discounted value will be?
- Another way to measure variability is the coefficient of variation (cv): For a random variable X, we have

$$cv(X) = \frac{\mathrm{Sd}(X)}{\mathbb{E}(X)}.$$

- We have $cv(V_5) = 0.0264, cv(V_{10}) = 0.0373, cv(V_{20}) = 0.0528$, which increase as n increases. Hence, we have more uncertainty in this perspective.

Moments of V_n: example B

Assume that $i_1, \dots, i_n$ are iid and $i_t \sim U(0.05, 0.09)$ for $t = 1, \dots, n$. Find the mean and variance of the present value of a unit payment in 20 periods.

Varying rate model: A_n and P_n

Recursive formula

- Unlike S_n and V_n, it is generally not easy to derive analytical expressions for the moments of A_n and P_n.
- For this, we can use recursive formulas of A_n and P_n to derive their moments.

Moments of A_n: mean

Proposition 2. *In the varying rate model, if $i_1, \dots, i_n$ have the same mean μ, then*

$$\mathbb{E}[A_n] = \frac{(1+\mu)^n - 1}{d},$$

where $d = \mu/(1+\mu)$.

Moments of A_n: second moment

Since

$$A_n^2 = (1+i_n)^2(A_{n-1}+1)^2,$$

and $i_1, \dots, i_n$ are independent, we have

$$\mathbb{E}[A_n^2] = \mathbb{E}[(i_n^2 + 2i_n + 1)]\mathbb{E}[(A_{n-1}^2 + 2A_{n-1} + 1)].$$

If $i_1, \dots, i_n$ have the same mean μ and variance σ^2, we get a recursive relation

$$\mathbb{E}[A_n^2] = (1 + 2\mu + \mu^2 + \sigma^2)(\mathbb{E}[A_{n-1}^2] + 2\mathbb{E}[A_{n-1}] + 1),$$

where $\mathbb{E}[A_{n-1}]$ is known.

Moments of A_n: example

Suppose that in a varying rate model $i_1, \dots, i_n$ follow a uniform distribution on $[0.02, 0.06]$. Find the mean and variance of A_5.

We first note that for $t = 1, \dots, n$,

$$\mu = \mathbb{E}[i_t] = \frac{0.02 + 0.06}{2} = 0.04,$$

and

$$\sigma^2 = \mathrm{Var}(i_t) = \frac{(0.06 - 0.02)^2}{12} \approx 0.00013333.$$

The mean of A_5 is

$$\mathbb{E}[A_5] = \frac{(1+\mu)^5 - 1}{d} = \frac{1.04^5 - 1}{0.04/1.04} = 5.63298.$$

Moments of A_n: example

To get the variance of A_5, we need to use the recursive formula

$$\begin{aligned}\mathbb{E}[A_n^2] &= (1+2\mu+\mu^2+\sigma^2)(\mathbb{E}[A_{n-1}^2]+2\mathbb{E}[A_{n-1}]+1)\\ &= 1.08173333(\mathbb{E}[A_{n-1}^2]+2\mathbb{E}[A_{n-1}]+1).\end{aligned}$$

We first compute

$$\mathbb{E}[A_1] = 1.04, \mathbb{E}[A_2] = 2.1216, \mathbb{E}[A_3] = 3.24646,$$

$$\mathbb{E}[A_4] = 4.41632.$$

Moments of S_n and A_n: example IV

Using the recursive relation, we get

$$\mathbb{E}[A_1^2] = 1.08173, \mathbb{E}[A_2^2] = 4.50189, \mathbb{E}[A_3^2] = 10.54158,$$

$$\mathbb{E}[A_4^2] = 19.50853, \mathbb{E}[A_5^2] = 31.73933.$$

Hence, the variance is

$$\mathrm{Var}(A_5) = 31.73933 - 5.63298^2 = 0.0089172.$$

Moments for P_n I

We can use similar techniques to derive moments for P_n as well. We have

$$\begin{aligned}P_n &= \sum_{t=1}^{n}\prod_{k=1}^{t}(1+i_k)^{-1}\\ &= \prod_{k=1}^{1}(1+i_k)^{-1} + \prod_{k=1}^{2}(1+i_k)^{-1} + \cdots + \prod_{k=1}^{n}(1+i_k)^{-1}\\ &= (1+i_1)^{-1}\left(1+(1+i_2)^{-1}+\cdots+\prod_{k=2}^{n}(1+i_k)^{-1}\right)\\ &= (1+i_1)^{-1}(1+P_{n-1}^*),\end{aligned}$$

where P_{n-1}^* is the time-1 value of payments of 1 at times $2,\dots,n$.

Moments for P_n II

Note that P_{n-1}^* and P_{n-1} have the same distribution for iid rates. Hence,

$$\begin{aligned}\mathbb{E}(P_n) &= \mathbb{E}((1+i_1)^{-1}(1+P_{n-1}^*))\\ &= \mathbb{E}((1+i_1)^{-1})(1+\mathbb{E}(P_{n-1}^*))\\ &= \mathbb{E}((1+i_1)^{-1})(1+\mathbb{E}(P_{n-1})),\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}(P_n^2) &= \mathbb{E}((1+i_1)^{-2}(1+P_{n-1}^*)^2)\\ &= \mathbb{E}((1+i_1)^{-2})(1+2\mathbb{E}(P_{n-1})+\mathbb{E}(P_{n-1}^2)).\end{aligned}$$

Infinite series of payments

Suppose that returns $i_1, i_2, i_3 \dots$ are iid. The time-0 value of payments of 1 at times $0, 1, 2, \dots$ is denoted by c_∞. Find the first two moments of c_∞.

Infinite series of payments

Moments for P_n: analytical solutions

Suppose $i_1, \dots, i_n$ are iid. Denote by u_k the kth moment $\mathbb{E}\left((1+i_t)^{-k}\right)$. Find the moments of P_n.

We have

$$\begin{aligned}
\mathbb{E}(P_n) &= \mathbb{E}\left(\sum_{t=1}^{n}\prod_{k=1}^{t}(1+i_k)^{-1}\right) \\
&= \sum_{t=1}^{n}\prod_{k=1}^{t}\mathbb{E}\left((1+i_k)^{-1}\right) \\
&= \sum_{t=1}^{n} u_1^t = \frac{u_1(1-u_1^n)}{1-u_1}.
\end{aligned}$$

Moments for P_n: analytical solutions

Finding the second moment is more difficult. We look at the case when $n = 2$: We have

$$\begin{aligned}
\mathbb{E}(P_2^2) &= \mathbb{E}\left(\left((1+i_1)^{-1} + (1+i_1)^{-1}(1+i_2)^{-1}\right)^2\right) \\
&= \mathbb{E}\left((1+i_1)^{-2}\right) + 2\mathbb{E}\left((1+i_1)^{-2}\right)\mathbb{E}\left((1+i_2)^{-1}\right) \\
&\quad + \mathbb{E}\left((1+i_1)^{-2}\right)\mathbb{E}\left((1+i_2)^{-2}\right) \\
&= u_2 + 2u_1u_2 + u_2^2.
\end{aligned}$$

Moments for P_n: analytical solutions

Next, we find the second moment of P_n. We have

$$\begin{aligned}
\mathbb{E}(P_n^2) &= \mathbb{E}\left(\left(\sum_{t=1}^{n}\prod_{k=1}^{t}(1+i_k)^{-1}\right)^2\right) \\
&= \mathbb{E}\left(\left(\sum_{t=1}^{n} V_t\right)^2\right) \\
&= \mathbb{E}\left(\sum_{t=1}^{n} V_t^2 + 2\sum_{j<k} V_jV_k\right) \\
&= \sum_{t=1}^{n}\mathbb{E}\left(V_t^2\right) + 2\sum_{j<k}\mathbb{E}\left(V_jV_k\right).
\end{aligned}$$

We know

$$\sum_{t=1}^{n}\mathbb{E}\left(V_t^2\right) = \sum_{t=1}^{n} u_2^t = \frac{u_2(1-u_2^n)}{1-u_2}.$$

Moments for P_n: analytical solutions

The rest is to find the expectation of the cross term. For $j < k$, we have

Moments for P_n: analytical solutions

Then

$$\begin{aligned}
\sum_{j<k}\mathbb{E}(V_jV_k) &= \sum_{j=1}^{n-1}\sum_{k=j+1}^{n} u_2^j u_1^{k-j} \\
&= \sum_{j=1}^{n-1}\left(\frac{u_2}{u_1}\right)^j \sum_{k=j+1}^{n} u_1^k \\
&= \sum_{j=1}^{n-1}\left(\frac{u_2}{u_1}\right)^j \frac{u_1^{j+1} - u_1^{n+1}}{1-u_1} \\
&= \frac{u_1}{1-u_1}\left(\sum_{j=1}^{n-1} u_2^j - u_1^n \sum_{j=1}^{n-1}\left(\frac{u_2}{u_1}\right)^j\right) \\
&= \frac{u_1u_2}{1-u_1}\left(\frac{1-u_2^{n-1}}{1-u_2} - u_1\frac{u_1^{n-1} - u_2^{n-1}}{u1 - u2}\right).
\end{aligned}$$