M9 Ruin Probability with Reinsurance

Topics in Insurance, Risk, and Finance

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Learning outcomes

- Find optimal reinsurance contracts by minimising ruin probabilities, in the classes of proportion reinsurance and excess of loss reinsurance, respectively.
- Know the optimal type of reinsurance contract given a budget constraint.

Ruin and reinsurance I

We have discussed how reinsurance affects insurers in a fixed period of time using the EUT. Next, we study the case of classical risk process using the ruin theory.

Given a compound Poisson aggregate claims process, the surplus claim process of an insurer **without reinsurance** is

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i,$$

where

- u is the initial surplus
- c is the premium income per unit of time
- $N(t)$ is the number of claims before or at time t
- X_i is the ith claim without reinsurance.

Ruin and reinsurance II

With reinsurance, the surplus claim process will become

$$U^*(t) = u + c^*t - \sum_{i=1}^{N(t)} X_i^*,$$

where

- u is the initial surplus
- c^* is the premium income per unit of time, net of reinsurance
- $N(t)$ is the number of claims before time t
- X_i^* is the ith claim, net of reinsurance.

Here, we make an implicit assumption that the **premium is paid to reinsurer continuously**.

Ruin and reinsurance III

- We are still in the classical risk model with reinsurance so the previous results can be applied here.
- Since it is generally hard to obtain an analytic expression of the ultimate ruin probability, we focus on studying the adjustment coefficient. With reinsurance, the adjustment coefficient R^* is given by

$$\lambda + c^* R^* = \lambda \mathbb{E}[\exp(R^* X_1^*)]$$

- Lundberg's inequality is used to approximate the ultimate ruin probability.
- We say a reinsurance arrangement is **optimal** in the sense that it maximizes the adjustment coefficient (hence minimises the approximated ultimate ruin probability).

Ruin and reinsurance IV

For the rest of the lecture, we make the assumptions below:

- Let X be the individual claim with $X \sim F$, $F(0) = 0$, and f the density.
- The loading factors of insurer and reinsurer are θ and θ_R.
- The Poisson parameter is λ.
- h is a reinsurance arrangement, i.e., the insurer pays $h(X) \leqslant X$ under the reinsurance arrangement. For instance,
  - $h(x) = \alpha x$ gives the proportional reinsurance
  - $h(x) = \min(x, M)$ gives the excess of loss reinsurance.

Then the insurer's premium rate, net of reinsurance, is given by,

$$c^* = (1+\theta)\lambda\mathbb{E}[X] - (1+\theta_R)\lambda\mathbb{E}[X - h(X)]$$

with $c^* > \lambda\mathbb{E}[h(X)]$.

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Proportional reinsurance: premium income I

- Assume that the retained proportion is $\alpha \in [0, 1]$.
- Following our previous assumption, the premium rate, before reinsurance, is

$$(1 + \theta)\lambda m_1.$$

- Similarly, the premium rate for the reinsurer is

$$(1 + \theta_R)\lambda(1 - \alpha)m_1.$$

- Consequently, the premium income for the insurer per unit of time, after reinsurance, is

$$\begin{aligned} c^* &= (1 + \theta)\lambda m_1 - (1 + \theta_R)\lambda(1 - \alpha)m_1 \\ &= ((1 + \theta) - (1 + \theta_R)(1 - \alpha))\lambda m_1. \end{aligned}$$

Proportional reinsurance: premium income II

Constraint one: After reinsurance, the expected claim for the insurer is αm_1. Therefore, we require the premium income per unit time, net of reinsurance, exceeds aggregate claims per unit time, i.e., $c^* \geqslant \lambda \alpha m_1$ and we get

$$(1+\theta) - (1+\theta_R)(1-\alpha) \geqslant \alpha,$$

which further gives

$$\alpha \geqslant 1 - \frac{\theta}{\theta_R}$$

Proportional reinsurance: premium income IV

Constraint two: We assume that the insurer pays for reinsurance out of the premium income it receives, i.e., $c^* \geqslant 0$

$$(1+\theta)\lambda m_1 \geqslant (1+\theta_R)\lambda(1-\alpha)m_1,$$

which gives

$$\alpha \geqslant \frac{\theta_R - \theta}{1+\theta_R}.$$

Proportional reinsurance: premium income III

If not specified, we always assume $\theta_R \geqslant \theta$. With this condition, we can see that

$$\alpha \geqslant \frac{\theta_R - \theta}{1 + \theta_R},$$

is implied by

$$\alpha \geqslant 1 - \frac{\theta}{\theta_R}.$$

Hence, Constraint one is crucial and it is sufficient to only consider it with the assumption that $\theta_R \geqslant \theta$.

Proportional reinsurance: example

Suppose that the individual claim amount X follows the exponential distribution $F(x) = 1 - e^{-x}$. Recall that $M_X(r) = 1/(1-r)$ for $r < 1$. Hence, we have

$$c^* = ((1+\theta) - (1+\theta_R)(1-\alpha))\lambda.$$

Then the adjustment coefficient can be calculated by

$$\lambda M_{X^*}(r) - \lambda - c^* r = 0,$$

where $M_{X^*}(r) = 1/(1-\alpha r)$.

We will consider the following three cases:

- (i) $\theta = \theta_R = 0.2$
- (ii) $\theta = 0.2$ and $\theta_R = 0.25$
- (iii) $\theta = 0.05$ and $\theta_R = 0.25$

Proportional reinsurance: example

(i) $\theta = \theta_R = 0.2$

Proportional reinsurance: example

(ii) $\theta = 0.2$ and $\theta_R = 0.25$

Proportional reinsurance: example

(iii) $\theta = 0.05$ and $\theta_R = 0.25$

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Excess of loss reinsurance: premium income

- Let M be the retention level.
- The premium rate for the insurer, before reinsurance, is

$$(1+\theta)\lambda m_1.$$

- The premium rate for the reinsurer is

$$(1+\theta_R)\lambda\mathbb{E}[\max(X-M,0)].$$

- Then the premium rate, net of reinsurance, is

$$c^* = (1+\theta)\lambda m_1 - (1+\theta_R)\lambda\mathbb{E}[\max(X-M,0)].$$

Excess of loss reinsurance: example

- The adjustment coefficient can be solved by

$$\lambda + c^* r = \lambda \left(\int_0^M \exp(rx) f(x) dx + \exp(rM)(1 - F(M)) \right).$$

- Assume that the individual claim follows $F(x) = 1 - e^{-x}$, $\theta = 0.1$, $\theta_R = 0.2$.

Excess of loss reinsurance: example

Constraint one: the premium income per unit time, net of reinsurance, exceeds aggregate claims per unit time:

$$
\begin{aligned}
c^* &= (1+\theta)\lambda m_1 - (1+\theta_R)\lambda \mathbb{E}[\max(X-M,0)] \\
&= 1.1\lambda - 1.2\lambda e^{-M} \\
&\geqslant \lambda \mathbb{E}[\min(X,M)] = \lambda(1-e^{-M}),
\end{aligned}
$$

which gives $M \geqslant \log 2 = 0.693$.

Excess of loss reinsurance: example

Constraint two: We assume that the insurer pays for reinsurance out of the premium income it receives:

$$1.1\lambda \geqslant 1.2\lambda \int_M^\infty (x - M)e^{-x}dx = 1.2\lambda e^{-M},$$

which gives $M \geqslant \log(12/11) = 0.0870$.

Hence, in this example it is sufficient to consider Constraint one, as in the case of proportional reinsurance.

Excess of loss reinsurance: example

The equation for the adjustment coefficient:

Excess of loss reinsurance: upper bound

- Remember an upper bound of adjustment coefficient without reinsurance is
$$R \leqslant \frac{2(c - \lambda m_1)}{\lambda m_2},$$
where c, m_1, m_2 are the premium income per unit of time, the first moment and the second moment of the individual claim.
- This formula can be still applied with reinsurance. However, c, m_1 and m_2 changes with reinsurance.
- We continue with the previous example with excess of loss reinsurance and compute an upper bound of R. That is, we need to compute c^*, m_1^*, and m_2^* which denote the corresponding component with reinsurance.

Excess of loss reinsurance: upper bound

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The optimal type of reinsurance

Consider the following question:

- An insurer can choose from different types of reinsurance contracts.
- It has a budget constraint on the reinsurance premium rate $c^* = C$, $C \in \mathbb{R}$.
- What type of reinsurance contract should be considered?

The optimal type of reinsurance

Theorem

Consider an excess of loss reinsurance with retention M and another reinsurance arrangement h. Assume that

- $\mathbb{E}[\min(X, M)] = \mathbb{E}[h(X)]$ *where M is the retention level of the excess of loss reinsurance*

*Then the **excess of loss reinsurance always dominates h** in the sense that its adjustment coefficient is larger.*

Note that the cost of reinsurance is always the same under the above assumption.

Proof

Proof

Proof