M7 Optimal Reinsurance with Expected Utility

Topics in Insurance, Risk, and Finance

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Introduction

Learning outcomes

- Define the collective risk model and the compound Poisson distribution.
- Know and derive properties of the compound Poisson distribution.
- Simulate from the compound Poisson distribution.
- Define reinsurance, proportion reinsurance, and excess of loss reinsurance.
- Find the optimal reinsurance contract using the expected utility theory.

Insurance portfolio

- In many insurance applications, portfolio losses can be simplified to the following two models.
- Denote by S the **aggregate risk** in a fixed period. Let $X_1, X_2, \ldots$ be a sequence of iid individual claims. We have
  - Individual risk model:
  $$S = \sum_{i=1}^{n} X_i,$$
  where n is a constant.
  - **Collective risk model**:
  $$S = \sum_{i=1}^{N} X_i,$$
  where N is a random number.

  We will cover some aspects of these models; advanced topics will be covered by another subject.

Reinsurance

- A **reinsurance contract** is an agreement in which one party (the reinsurer) agrees to indemnify another party (the insurer) for parts of its insurance risk.
- Reinsurance thus can be seen as a particular form of insurance.
- Why reinsurance?
- There are many reasons: prevention of extreme losses, reduction of capital requirements, increasing underwriting capacity and so on.

Two types of reinsurance: proportion reinsurance

Proportion reinsurance: the reinsurer covers a prespecified proportion of each risk in the portfolio and the reinsurance premium is in proportion to the risk ceded.

If the insurer has a **retained proportion** α, then when a loss X occurs, the insurer will need to pay αX and the reinsurer will pay $(1-\alpha)X$.

Two types of reinsurance: excess of loss reinsurance

Excess of loss reinsurance: the reinsurer pays the claim which is beyond a prespecified limit. In other words, the insurer's liability is capped. The cap is referred to as the **retention** of the insurer.

If the insurer has a retention limit M, then when a loss X occurs, the insurer will need to pay $\min(X, M)$ and the reinsurer will pay $\max(X - M, 0)$.

Optimal reinsurance

- Given the various choices of reinsurance products, which is the optimal one for the insurer?
- We will study the insurer's aggregate risk after reinsurance in the framework of the **expected utility theory**.

Compound Poisson distribution

Compound Poisson distribution

Let $X_1, X_2, ...$ be a sequence of iid individual claims and N be the number of claims, independent of all X in a fixed period. The portfolio loss S below is called the **collective risk model**:

$$S = \sum_{i=1}^{N} X_i.$$

By convention, $S = 0$ when $N = 0$.

- In the special case where N follows the Poisson distribution, S is said to follow the **compound Poisson distribution**.
- We will assume that the aggregate risk follows the compound Poisson distribution.

Recap: Poisson distribution

A random variable X is said to follow the **Poisson distribution** with parameter λ, denoted by $X \sim Poisson(\lambda)$, if

$$\mathbb{P}(X = x) = \frac{e^{-\lambda}\lambda^x}{x!},$$

where $x = 0, 1, 2, ...$

Properties of the Poisson distribution:

- $\mathbb{E}(X) = \lambda$
- $\mathrm{Var}(X) = \lambda$
- $M_X(t) = \mathbb{E}(\exp(tX)) = \exp(\lambda(e^t - 1))$

Recap: mgf

We have

$$\begin{aligned} M_X(t) &= \mathbb{E}(\exp(tX)) \\ &= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \\ &= e^{-\lambda} \exp(\lambda e^t) \\ &= \exp(\lambda(e^t - 1)). \end{aligned}$$

Properties of compound Poisson distributions

Notations:

- $X_i \sim F$ with density f such that $F(0) = 0$ (i.e., claims are positive).
- $m_k = \mathbb{E}(X_i^k)$, $k = 1, 2, 3, ...$
- $M_X(r) = \mathbb{E}(\exp(rX_i))$

Proposition 1. *Suppose that S follows the compound Poisson distribution with aforementioned assumptions, then*

- $\mathbb{E}(S) = \lambda m_1$, - $\mathrm{Var}(S) = \lambda m_2$, - $M_S(r) = \exp(\lambda(M_X(r) - 1))$.

Proof: expectation

Hint: $\mathbb{E}(S) = \mathbb{E}(\mathbb{E}(S|N))$

Proof: variance

Hint: $\text{Var}(S) = \mathbb{E}(\text{Var}(S|N)) + \text{Var}(\mathbb{E}(S|N))$

Proof: moment generating function

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Proof:

Example: normal approximation

Suppose that the claim for annual policy i is $X_i \sim N(0.7P, 4P^2)$, where $P = 5000$ is the annual premium. Claims are independent. Let the initial wealth be 0.1 million. The number of claims follows $Poisson(100)$, independent of claim sizes. Find the probability that the insurer's wealth is below 0 at the end of the year assuming 100 policies were sold at the start of the year. Comment on the model assumption.

Example: normal approximation

Optimal reinsurance with expected utility

Optimal reinsurance

- Insurers pay premiums to reinsurers to transfer part of their losses.
- Reinsurance can reduce the variability of the aggregate claims or the probability of ruin.
- A reinsurance contract is said to be **optimal** if the insurer's expected utility is maximised or the probability of ruin is minimised.

An expected utility model

- An insurer has utility function u and wealth random variable X.
- **Goal**: maximise $\mathbb{E}(u(X))$
- The insurer is **risk-averse**: u is increasing and concave.
- Risk aversion means: (a) the more wealth the better (b) the marginal utility is decreasing.
- We will study how reinsurance can affect an insurer's decision making, i.e., how to make the optimal decision in the presence of reinsurance.
- Q: What are possible limitations of this decision model?

Utility function

We make the following **assumptions**:

- The **utility function** u is an exponential utility function:

$$u(x) = -\exp(-\beta x),$$

  where $\beta > 0$. This implies that the insurer is risk-averse.

- The **aggregate risk** S follows the compound Poisson distribution with Poisson parameter λ and the individual claim distribution is F with density f.

- The **wealth random variable** after reinsurance is

$$X = w + p - P_R - S_I,$$

  where

  - w: initial wealth
  - p: insurance premium
  - P_R: reinsurance premium
  - S_I: aggregate risk net of reinsurance

Goal

The goal is to maximise the expected utility of the insurer:

$$\begin{aligned}\max \mathbb{E}[u(X)] &= \max \mathbb{E}[u(w+p-P_R-S_I)] \\ &= \max \exp(-\beta(w+p))(-\exp(\beta P_R))\mathbb{E}[\exp(\beta S_I)].\end{aligned}$$

Since w and p are constant, the above problem is equivalent to:

$$\max(-\exp(\beta P_R))\mathbb{E}[\exp(\beta S_I)]$$

We need to decide P_R and S_I such that the above expression can be written explicitly.

Proportion reinsurance

Assumes the reinsurer covers $1-\alpha$ of each claim and the reinsurance premium is calculated by the exponential principle with parameter A (i.e., for a random loss X, $P_R = \log \mathbb{E}[\exp(AX)]/A = \log M_X(A)/A$).

Let us break down the optimisation problem:

- Write down S_I and P_R
- Write down the objective function $(-\exp(\beta P_R))\mathbb{E}[\exp(\beta S_I)]$
- Find α^* which maximises $(-\exp(\beta P_R))\mathbb{E}[\exp(\beta S_I)]$

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Excess of loss reinsurance

Let us now assume that the insurer effects excess of loss reinsurance with retention level M and that the reinsurance premium is calculated by the expected value principle with loading θ (i.e., for a random loss X, $P_R = (1+\theta)\mathbb{E}[X]$).

Let us break down the optimisation problem:

- Write down S_I and P_R
- Write down the objective function $(-\exp(\beta P_R))\mathbb{E}[\exp(\beta S_I)]$
- Find M^* which maximises $(-\exp(\beta P_R))\mathbb{E}[\exp(\beta S_I)]$

Excess of loss reinsurance

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