M11 Ruin probability and reinsurance

Topics in Insurance, Risk, and Finance

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2024

Adjustment Coefficient and Lundberg's inequality

Ruin probability for exponential claims

Theorem 1. Given the classical risk process, if claims $X_1, X_2, ...$ follow an exponential distribution $F(x) = 1 - e^{-\alpha x}$ for $x \geqslant 0$,

$$\psi(u) = \psi(0)\exp(-Ru),$$

where $R = \alpha - \lambda/c$ and $\psi(0) = \lambda/(\alpha c)$.

Example: exponential claims

Suppose $X_1, X_2, ...$ follow an exponential distribution $F(x) = 1 - e^{-\alpha x}$ for $x \geqslant 0$ and $c = (1+\theta)\lambda m_1$. What is $\psi(u)$?

We have

$$\begin{aligned}\psi(u) &= \psi(0)\exp(-Ru)\\ &= \frac{\lambda}{\alpha c}\exp(-(\alpha - \lambda/c)u)\\ &= \frac{1}{1+\theta}\exp\left(-\frac{\alpha\theta u}{1+\theta}\right).\end{aligned}$$

Hence, the ultimate ruin probability is independent of the Poisson parameter λ.

Lundberg's inequality

As in the discrete time risk model, an upper bound also exists for the ultimate ruin probability in the classical risk model.

Theorem 2. Given the classical risk process, the ultimate ruin probability has an upper bound given by

$$\psi(u) \leqslant \exp(-Ru),$$

where R is the **adjustment coefficient** which is the unique positive root of the equation below **

$$\lambda M_X(R) - \lambda - cR = 0.$$

**

- If $c = (1+\theta)\lambda m_1$, the above equation reduces to

  $$M_X(R) = 1 + (1+\theta)m_1 R,$$

  so that R is independent of the Poisson parameter λ.

Additional comments

- For large initial surplus u, the ultimate ruin probability is close to the upper bound. Hence, we have the approximation

  $$\psi(u) \approx \exp(-Ru),$$

  which is often used in actuarial literature.

- Clearly, the upper bound $\exp(-Ru)$ decreases as R increases, where $\exp(-Ru)$ is used as an approximation of $\psi(u)$. Arguably, the ultimate ruin probability $\psi(u)$ also decreases as R increases.
- By the second bullet point, we can regard the adjustment coefficient R as a (reverse) measure of risk for insurers: The larger R is, the less risk insurers face.
- e^{-R} can be regarded as the factor by which the ruin probability decreases given a unit increase in the initial surplus.

Proof of Lundberg's inequality I

- Let $\psi_n(u)$, $n = 1, 2, ...$, be the probability that ruin happens before nth claim.
- Note that $\lim_{n\to\infty} \psi_n(u) = \psi(u)$ and $\psi_n(u)$ increases as n increases.
- Hence it suffices to show $\psi_n(u) \leqslant \exp(-Ru)$ for all $n = 1, 2, ...$ This is done by induction.

Proof of Lundberg's inequality II

Assuming that $\psi_n(u) \leqslant \exp(-Ru)$, we show that $\psi_{n+1}(u) \leqslant \exp(-Ru)$ holds.

$$\begin{aligned}
\psi_{n+1}(u) &= \int_0^\infty \lambda \exp(-\lambda t) \int_{u+ct}^\infty f(x)\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^\infty \lambda \exp(-\lambda t) \int_0^{u+ct} \psi_n(u+ct-x) f(x)\mathrm{d}x\mathrm{d}t \\
&\leqslant \int_0^\infty \lambda \exp(-\lambda t) \int_{u+ct}^\infty f(x)\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^\infty \lambda \exp(-\lambda t) \int_0^{u+ct} \exp(-R(u+ct-x)) f(x)\mathrm{d}x\mathrm{d}t \\
&\leqslant \int_0^\infty \lambda \exp(-\lambda t) \int_0^\infty \exp(-R(u+ct-x)) f(x)\mathrm{d}x\mathrm{d}t = \exp(-Ru).
\end{aligned}$$

The last equality uses the fact that $\lambda M_X(R) = \lambda + cR$.

Proof of Lundberg's inequality III

The rest is to show $\psi_1(u) \leqslant \exp(-Ru)$. We have

$$\begin{aligned}
\psi_1(u) &= \int_0^\infty \lambda e^{-\lambda t} \int_{u+ct}^\infty f(x)dxdt \\
&\leqslant \int_0^\infty \lambda e^{-\lambda t} \int_{u+ct}^\infty f(x)\exp(-R(u+ct-x))dxdt \\
&\leqslant \int_0^\infty \lambda e^{-\lambda t} \int_0^\infty f(x)\exp(-R(u+ct-x))dxdt \\
&= \exp(-Ru).
\end{aligned}$$

The proof is done.

Uniqueness of the root I

To show $\lambda M_X(R) - \lambda - cR = 0$ has a unique root, we need the assumption that for some $\gamma \leqslant \infty$, $M_X(r)$ is finite for all $r < \gamma$ and $\lim_{r\to\gamma} M_X(r) = \infty$.

Define

$$g(r) = \lambda M_X(r) - \lambda - cr.$$

Note that $g(0) = 0$. We first show $\lim_{r\to\gamma} g(r) = \infty$.

For $\gamma < \infty$, it is obvious.

Uniqueness of the root II

For $\gamma = \infty$, note there exists $\epsilon > 0$ such that

$$\begin{aligned}
M_X(r) &= \int_0^\infty e^{rx} f(x)\mathrm{d}x \geqslant \int_\epsilon^\infty e^{rx} f(x)\mathrm{d}x \\
&\geqslant \int_\epsilon^\infty e^{r\epsilon} f(x)\mathrm{d}x = e^{r\epsilon} P(X \geqslant \epsilon).
\end{aligned}$$

Hence,

$$\lim_{r\to\gamma} g(r) \geqslant \lim_{r\to\gamma} (\lambda e^{r\epsilon} P(X \geqslant \epsilon) - \lambda - cr) = \infty$$

Uniqueness of the root III

Take derivatives of $g(r)$. We get

$$\frac{\mathrm{d}}{\mathrm{d}r} g(r) = \lambda \frac{\mathrm{d}}{\mathrm{d}r} M_X(r) - c.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}r} g(r)|_{r=0} = \lambda m_1 - c < 0.$$

Also

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2} g(r) = \lambda \frac{\mathrm{d}^2}{\mathrm{d}r^2} M_X(r) > 0.$$

Consequently, $g(r)$ is a convex function with $g(0) = 0$ and $\lim_{r\to\gamma} g(r) = \infty$. The desired result is obtained.

Example: exponential losses I

If the individual claims follow the exponential distribution $F(x) = 1 - \exp(-\alpha x)$ for $x \geqslant 0$, then $M_X(r) = \alpha/(\alpha - r)$ for $r < \alpha$. Hence, we need to solve

$$\lambda M_X(R) = \lambda \frac{\alpha}{\alpha - R} = \lambda + cR,$$

which is equivalent to

$$cR^2 + (\lambda - \alpha c)R = 0.$$

Hence, $R = \alpha - \lambda/c$.

Example: exponential losses II

If we write $c = (1+\theta)\lambda m_1 = (1+\theta)\lambda/\alpha$, we have

$$R = \alpha - \frac{\lambda}{c} = \frac{\theta\alpha}{1+\theta}.$$

Since

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} = \frac{\alpha}{(1+\theta^2)} > 0,$$

- We can see that as θ increases, R increases, and essentially $\exp(-Ru)$ decreases which is the upper bound of $\psi(u)$.
- That makes sense as we are charging more premium at a higher θ.
- We obtained an explicit solution in the above example. In most cases, however, R can only be solved numerically.

Example: mixtures of exponential distributions I

If the individual claims follow the distribution

$$F(x) = 1 - 0.5(\exp(-3x) + \exp(-7x)),$$

for $x \geqslant 0$, find R when $\lambda = 3$ and $c = 1$.

Example: mixtures of exponential distributions II

The moment generating function of individual claim is

$$M_X(r) = 0.5\left(\frac{3}{3-r} + \frac{7}{7-r}\right).$$

Note that $M_X(r)$ exists for $r < 3$.Hence, by $\lambda + cR = \lambda M_X(R)$, we get

$$R^3 - 7R^2 + 6R = R(R-1)(R-6) = 0,$$

which gives $R = 1$.

An upper bound of adjustment coefficient

Proposition 1. *In the classical risk process model, we have*

$$R < \frac{2(c - \lambda m_1)}{\lambda m_2}.$$

- If R is small, this upper bound can a good approximation of R.
- If $c = (1+\theta)\lambda m_1$, we get $R < 2\theta m_1/m_2$.

An upper bound of adjustment coefficient: proof

We have

$$\exp(x) = 1 + x + \frac{x^2}{2} + o(x) > 1 + x + \frac{x^2}{2}.$$

Hence we have

$$\begin{aligned}\lambda + cR = \lambda M_X(R) &= \lambda\mathbb{E}(\exp(RX)) \\ &= \lambda\mathbb{E}\left(1 + RX + \frac{(RX)^2}{2} + o(RX)\right) \\ &> \lambda\mathbb{E}\left(1 + RX + \frac{(RX)^2}{2}\right) = \lambda\left(1 + Rm_1 + \frac{R^2}{2}m_2\right).\end{aligned}$$

Solving the above inequality, we get the desired result.

Example: mixtures of exponential distributions (upper bound of R)

If the individual claims follow the exponential distribution $F(x) = 1 - 0.5(\exp(-3x) - \exp(-7x))$ for $x \geqslant 0$, we have seen that $R = 1$ when $\lambda = 3$ and $c = 1$.

We have

$$m_1 = 0.5\left(\frac{1}{3} + \frac{1}{7}\right) = \frac{5}{21},$$

and

$$m_2 = 0.5\left(\frac{2}{3^2} + \frac{2}{7^2}\right) = \frac{58}{441},$$

The upper bound is

$$R < \frac{2(c - \lambda m_1)}{\lambda m_2} = 1.45,$$

which is not close to the true value.

A lower bound of adjustment coefficient

Proposition 2. *In the classical risk process model, if $X_i \leqslant M$ where $M > 0$, we have*

$$R > \frac{1}{M}\log\left(\frac{c}{\lambda m_1}\right).$$

Hence if each individual claim has an upper bound, we can also derive a lower bound for R.

A lower bound of adjustment coefficient: proof

Assume that $X \leqslant M$ where $M > 0$. We first show for $0 \leqslant x \leqslant M$:

$$\exp(Rx) \leqslant \frac{x}{M}\exp(RM) + 1 - \frac{x}{M}.$$

We have

$$\begin{aligned}\frac{x}{M}\exp(RM) + 1 - \frac{x}{M} &= \frac{x}{M}\sum_{j=0}^{\infty}\frac{(RM)^j}{j!} + 1 - \frac{x}{M} \\ &= 1 + \sum_{j=1}^{\infty}\frac{R^jM^{j-1}x}{j!} \\ &\geqslant 1 + \sum_{j=1}^{\infty}\frac{(Rx)^j}{j!} = \exp(Rx).\end{aligned}$$

A lower bound of adjustment coefficient: proof

We have

$$\begin{aligned}\lambda + cR = \lambda M_X(R) &= \lambda\int_0^{\infty}\exp(Rx)f(x)\mathrm{d}x \\ &\leqslant \lambda\int_0^{\infty}\left(\frac{x}{M}\exp(RM) + 1 - \frac{x}{M}\right)f(x)\mathrm{d}x \\ &= \frac{\lambda}{M}\exp(RM)m_1 + \lambda - \frac{\lambda}{M}m_1.\end{aligned}$$

Then

$$\frac{c}{\lambda m_1} \leqslant \frac{1}{RM}(\exp(RM)-1) = 1 + \frac{RM}{2!} + \frac{(RM)^2}{3!} + \dots$$
$$\leqslant 1 + \frac{RM}{1!} + \frac{(RM)^2}{2!} + \cdots = \exp(RM).$$

Hence

$$R > \frac{1}{M}\log\left(\frac{c}{\lambda m_1}\right).$$

Ruin probabilities against changing parameters

Ruin probabilities

Recall that in the classical risk process, continuous-time ruin probabilities are defined as

$$\psi(u) = P(T < \infty),$$

and

$$\psi(u,t) = P(T < t).$$

where $T = \min\{t > 0 : U(t) < 0\}$ is the first time of ruin.

Ruin probabilities against t

For $0 < t_1 \leqslant t_2 < \infty$, and $u \geqslant 0$,

$$\psi(u,t_1) \leqslant \psi(u,t_2) \leqslant \psi(u).$$

This is clear as

$$\{T < t_1\} \subseteq \{T < t_2\} \subseteq \{T < \infty\}.$$

Hence $\psi(u,t)$ is an increasing function of t.

Ruin probabilities against u

For $0 \leqslant u_1 \leqslant u_2$,

$$\psi(u_1) \geqslant \psi(u_2).$$

This is because

$$\psi(u) = \mathbb{P}(u + ct - S(t) < 0 \quad \text{for some} \quad t > 0)$$

and that

$$\{u_2 + ct - S(t) < 0 \quad \text{for some} \quad t > 0\} \subseteq \{u_1 + ct - S(t) < 0 \quad \text{for some} \quad t > 0\}$$

Hence, $\psi(u)$ is a decreasing function of u. Similarly, we have

$$\psi(u_1,t) \geqslant \psi(u_2,t).$$

Ruin probabilities against θ

- Probabilities $\psi(u)$ and $\psi(u,t)$ are both decreasing against θ.
- This is intuitively true as a larger θ means more premium income.
- One can also prove this result using similar arguments for the previous result regarding ruin probabilities against u.

Reinsurance and expected utility

Decision models

- Recall that for an agent/decision maker with utility function u and wealth random variable X, the agent's goal is to maximize $\mathbb{E}(u(X))$.
- The agent is **risk-averse** if its utility function is increasing and concave.
- Risk aversion means: (a) the more wealth the better

(b) the marginal utility is decreasing.

- We will study how reinsurance can affect an insurer's decision making, i.e., how to make the optimal decision in the presence of reinsurance.
- One can also use risk measures like VaR to measure the agent's risk (although not covered in this subject).

Reinsurance

- Insurers pay premiums to reinsurers to transfer part of their losses.
- Reinsurance reduces the variability of the aggregate claims so that the probability of ruin can be reduced.
- A reinsurance contract is said to be **optimal** if the insurer's utility is maximized or the probability of ruin is minimized.

Two types of reinsurance: proportion reinsurance

Proportion reinsurance: the reinsurer covers a prespecified proportion of each risk in the portfolio and the reinsurance premium is in proportion to the risk ceded.

If the insurer has a **retained proportion** α, then when a loss X occurs, the insurer will need to pay αX and the reinsurer will pay $(1-\alpha)X$.

Two types of reinsurance: excess of loss reinsurance

Excess of loss reinsurance: the reinsurer pays the claim which is beyond a prespecified limit. In other words, the insurer's liability is capped. The cap is referred to as the **retention** of the insurer.

If the insurer has a retention limit M, then when a loss X occurs, the insurer will need to pay $\min(X, M)$ and the reinsurer will pay $\max(X-M, 0)$.

Application of utility theory I

Throughout this section, we make the following *assumptions*:

- The insurer uses the **exponential utility function**:

$$u(x) = -\exp(-\beta x),$$

  where $\beta > 0$. This implies that the insurer is risk-averse.
- The insurer's claim number follows a Poisson distribution with Poisson parameter λ and the individual claim distribution is F with density f and $F(0) = 0$. This means that the aggregate claim follows a compound Poisson distribution.
- Note that this is not the classical risk model.

Application of utility theory II

Suppose that the insurer with policies is considering buying reinsurance. The insurer has wealth at the end of a period:

$$W_I = W + P - P_R - S_I,$$

where

- W is the insurer's wealth at the start of the period
- P is the premium the insurer receives to cover the risk
- P_R is the amount of the reinsurance premium
- S_I denotes the amount of claims paid by the insurer net of reinsurance

Application of utility theory III

The goal is to maximize the expected utility of the insurer:

$$\begin{aligned}\max \mathbb{E}[u(W_I)] &= \max \mathbb{E}[u(W + P - P_R - S_I)] \\ &= \max \exp(-\beta(W+P))(-\exp(\beta P_R))\mathbb{E}[\exp(\beta S_I)].\end{aligned}$$

Since W and P are constant, the above problem is equivalent to:

$$\max(-\exp(\beta P_R))\mathbb{E}[\exp(\beta S_I)].$$

We need to decide P_R and S_I such that the above expression can be maximized.

Two premium principles

- The **expected value principle** sets the premium of a claim X as

$$P_X = (1+\theta)\mathbb{E}[X],$$

  where $\theta > 0$ is the premium loading factor.

- For a utility function u and an insurer with initial wealth W, premium P_X and a claim X, the **principle of Zero utility** sets the premium by the following equality:

$$u(W) = \mathbb{E}[u(W + P_X - X)].$$

In the case of exponential utility $u(x) = -\exp(-\beta x)$ (called the exponential principle), we have

$$P_X = \frac{\log \mathbb{E}[\exp(\beta X)]}{\beta} = \frac{\log M_X(\beta)}{\beta}.$$

Application of utility: proportional reinsurance I

Assumes the reinsurer covers $1 - \alpha$ of each claim and the reinsurance premium is calculated by the exponential principle with parameter A.

The reinsurance premium is thus

$$P_R = \frac{\lambda}{A}\left(\int_0^\infty e^{(1-\alpha)Ax} f(x)dx - 1\right),$$

and

$$\mathbb{E}[\exp(\beta S_I)] = \exp\left(\lambda\left(\int_0^\infty e^{\alpha\beta x} f(x)dx - 1\right)\right).$$

Application of utility: proportional reinsurance II

Therefore, we are to maximize

$$\begin{aligned} &-\exp(\beta P_R)\mathbb{E}[\exp(\beta S_I)] \\ &= -\exp\left(\frac{\lambda\beta}{A}\left(\int_0^\infty e^{(1-\alpha)Ax} f(x)dx - 1\right) + \lambda\left(\int_0^\infty e^{\alpha\beta x} f(x)dx - 1\right)\right), \end{aligned}$$

which is equivalent to minimize

$$\begin{aligned} h(\alpha) &= \frac{\lambda\beta}{A}\int_0^\infty e^{(1-\alpha)Ax} f(x)dx + \lambda\int_0^\infty e^{\alpha\beta x} f(x)dx \\ &= \lambda\int_0^\infty \left(A^{-1}\beta e^{(1-\alpha)Ax} + e^{\alpha\beta x}\right) f(x)dx. \end{aligned}$$

Application of utility: proportional reinsurance III

Taking derivatives of $h(\alpha)$, we get

$$\frac{d}{d\alpha}h(\alpha) = \lambda\int_0^\infty \left(-x\beta e^{(1-\alpha)Ax} + \beta x e^{\alpha\beta x}\right) f(x)dx,$$

and

$$\frac{d^2}{d\alpha^2}h(\alpha) = \lambda\int_0^\infty \left(Ax^2\beta e^{(1-\alpha)Ax} + \beta^2 x^2 e^{\alpha\beta x}\right) f(x)dx > 0.$$

Hence, $h(\alpha)$ has a minimum at $\alpha = A/(A+\beta)$.

Application of utility: excess of loss reinsurance I

Let us now assume that the insurer effects excess of loss reinsurance with retention level M and that the reinsurance premium is calculated by the expected value principle with loading θ. The reinsurance premium is thus

$$P_R = (1+\theta)\lambda\int_M^\infty (x - M)f(x)dx,$$

and that

$$E[\exp(\beta S_I)] = \exp\left(\lambda\left(\int_0^M e^{\beta x} f(x)dx + e^{\beta M}(1 - F(M)) - 1\right)\right).$$

Application of utility: excess of loss reinsurance II

We are to maximize

$$-\exp(\beta P_R)\mathbb{E}[\exp(\beta S_I)].$$

Equivalently, we are to minimize

$$g(M) = (1+\theta)\lambda\beta\int_M^\infty (x - M)f(x)dx + \lambda\left(\int_0^M e^{\beta x} f(x)dx + e^{\beta M}(1 - F(M))\right).$$

Taking derivatives of $g(M)$, we have

$$\begin{aligned}\frac{d}{dM}g(M) &= (1+\theta)\lambda\beta(F(M)-1) + \lambda\beta e^{\beta M}(1-F(M)) \\ &= \lambda\beta(1-F(M))(e^{\beta M}-1-\theta),\end{aligned}$$

which equals to 0 when $M = \log(1+\theta)/\beta$. Since the second derivative of $g(M)$ is positive at $\log(1+\theta)/\beta$, we found the minimum of $g(M)$.

Example: excess of loss reinsurance

Aggregate claims from a risk have a compound Poisson distribution with Poisson parameter 100, and individual claim amounts are exponentially distributed with mean 100. The insurer of this risk decides to effect excess of loss reinsurance, and the reinsurance premium is calculated according to the variance principle with parameter 0.5 (i.e., for a random loss X, the premium is $\mathbb{E}(X) + 0.5\mathrm{Var}(X)$). Find the retention level that maximizes the insurer's expected utility of wealth with respect to the utility function $u(x) = -\exp(-0.005x)$.

Example: excess of loss reinsurance

- Let X_i be the ith claim after reinsurance.
- Let $Y_1, Y_2, ...$ be the losses of the reinsurer.
- We want to maximize $\mathbb{E}(u(W + P - P_R - \sum_{i=1}^N X_i))$.
- Essentially, we need to maximize

$$\mathbb{E}\left(u\left(-P_R - \sum_{i=1}^N X_i\right)\right) = -\exp(\beta P_R)\mathbb{E}\left(\exp\left(\beta\sum_{i=1}^N X_i\right)\right).$$

Example: excess of loss reinsurance

We have

$$\begin{aligned}P_R &= \mathbb{E}\left(\sum_{i=1}^N Y_i\right) + 0.5\mathrm{Var}\left(\sum_{i=1}^N Y_i\right) \\ &= 100\mathbb{E}(Y_i) + 0.5 * 100 * \mathbb{E}(Y_i^2).\end{aligned}$$

Denote by M the retention level. We have

$$\mathbb{E}(Y_i) = 100e^{-0.01M},$$

and

$$\mathbb{E}(Y_i^2) = 20000e^{-0.01M}.$$

Then

$$P_R = 1010000e^{-0.01M}.$$

Example: excess of loss reinsurance

In this question, $\beta = 0.005$. Then

$$\mathbb{E}\left(\exp\left(\beta\sum_{i=1}^N X_i\right)\right) = \exp(100(1-e^{-0.005M})).$$

Example: excess of loss reinsurance

We are then to maximize

$$-\exp(\beta P_R)\mathbb{E}(\exp(\beta\sum_{i=1}^N X_i)) = -\exp(\beta 1010000e^{-0.01M} + 100(1-e^{-0.005M})).$$

Let

$$h(M) = 5050e^{-0.01M} + 100(1-e^{-0.005M}).$$

Then

$$h'(M) = -50.5e^{-0.01M} + 0.5e^{-0.005M}.$$

The optimal retention level is

$$M = 200(\log 101).$$

Second derivative is positive at this point.